8 The Capital Asset Pricing Model

Although born in the 1960's, the Capital Asset Pricing Model (CAPM) has been widely used for decades in many aspects in finance, such as estimating the cost of capital for firms and evaluating the performance of portfolios. And it has been the most widely taught model in finance.

8.1 Derivation of the Model

The model assumes that investors are risk averse, and when they make investment decisions, they only care about the mean and the variance of their investments. Given a required level of expected return, rational investors choose the investment that minimizes the variance, which represents the risk, of the investment. Although overly simplified, this is at least a starting point of modeling investors' behaviors.

Now consider a market in which there are N risky assets, each with a random return $R_i, i = 1, 2, \ldots, N$, and the investors are not allowed to short-sell these risky assets. The problem a rational investor face is to choose weights $w_i, i = 1, 2, \ldots, N$ on these N assets so that given a required level of expected return μ, the variance of the return on the portfolio of assets is minimized. Suppose that the covariance matrix of the return vector $R = (R_1, R_2, \ldots, R_N)'$ is Ω, and let $\mu = \mathbb{E}R$, then we may formulate the investor's problem as

$$\begin{aligned} \max_{w} \quad & w'\Omega w \\ \text{s.t.} \quad & w'\mu = a \\ & w'\iota = 1, \end{aligned}$$

where $w = (w_1, w_2, \ldots, w_N)'$, $\iota = (1, 1, \ldots, 1)'$, and a is a positive number representing the required level of expected return.

We may solve the problem to obtain the optimal choice of the weights:

$$w = v_1 + a v_2$$

where

$$v_1 = \frac{1}{D}[B\Omega^{-1}\iota - A\Omega^{-1}\mu],$$

$$v_2 = \frac{1}{D}[C\Omega^{-1}\mu - A\Omega^{-1}\iota],$$

$A = \iota'\Omega^{-1}\mu, B = \mu'\Omega^{-1}\mu, C = \iota'\Omega^{-1}\iota$, and $D = BC - A^2$.

If we substitute this optimal w back into the objective function $w'\Omega w$, we get a relationship between the variance of the return on the portfolio and the level of expected return a. If we plot the level of expected return against the standard deviation of the return on the portfolio on the σ-r plane, we get the minimum variance frontier for risky assets.

Now suppose that the investors could borrow or lend without restrictions at a risk-free rate R_f. If an investor borrows or lends some money and use the rest to purchase a portfolio of risky assets that is represented by a point G on the σ-r plane, then the investor's position could be represented by a point on the ray that originates from 0 and passes through G. The only mean-variance efficient frontier with a riskless asset for this investor is the ray that originates from 0, is tangent to the minimum variance frontier for risky assets, and is upward sloping. Let the portfolio represented by the tangent point be M. This portfolio is the portfolio the investor would choose for his/her risky investments.

Another import assumption in the CAPM is that the expectation and the variance of the return R to assets are common knowledge to all participants in the market. This implies that every investor in the market knows the mean-variance efficient frontier and will choose to invest into the same portfolio M. Therefore, M becomes the market portfolio, and the ray that originates from 0 and passes through M becomes the Capital Market Line. We denote the return on the market portfolio by R_M, and the variance of the return on the market portfolio by σ_M^2. The slope of the Capital Market Line on the σ-r plane is given by

$$\frac{\mathbb{E}R_M - R_f}{\sigma_M}.$$

Now consider an investor who invests a fraction $(1-x)$ of wealth into the risky asset i and a fraction x of wealth into the market portfolio. The expected return of this portfolio is

$$\mathbb{E}r_p = x\mathbb{E}R_M + (1-x)\mathbb{E}R_i,$$

and the variance of the return of this portfolio is

$$\sigma_p^2 = x^2\sigma_M^2 + (1-x)^2\sigma_i^2 + 2x(1-x)\sigma_{iM}$$

where σ_i^2 is the variance of the return on asset i, and σ_{iM} is the covariance of the return on the market portfolio and the return on the asset i.

We may track the trajectory of the pair $(\sigma_p, \mathbb{E}r_p)$ on the σ-r plane as x changes. We know that this trajectory passes through M at $x = 1$ but cannot cross the Capital Market Line, and is smooth. This implies that the trajectory must be tangent to the Capital Market Line at M. Therefore,

$$\left.\frac{\mathrm{d}\mathbb{E}r_p}{\mathrm{d}\sigma_p}\right|_{x=1} = \left.\frac{\frac{\mathrm{d}\mathbb{E}r_p}{\mathrm{d}x}}{\frac{\mathrm{d}\sigma_p}{\mathrm{d}x}}\right|_{x=1} = \frac{\mathbb{E}R_M - R_f}{\sigma_M}.$$

Some algebra shows that

$$\mathbb{E}R_i - R_f = \beta_{iM}(\mathbb{E}R_M - R_f)$$

where

$$\beta_{iM} = \frac{\sigma_{iM}}{\sigma_M^2}.$$

This is the so-called Capital Asset Pricing Model, and β_{iM} is usually called the *beta* of the risky asset i.

If we write the excess returns as $Z = R - R_f$ and $Z_M = R_M - R_f$, the CAPM says that the expected excess return on all assets are linearly related their betas, and no other variables has marginal explanatory power. The betas measure the sensitivity of the asset returns to variation in the market return.

8.2 Testing the CAPM

Now suppose we have multiple periods. We use time subscripts to denote variables corresponding to different time periods. If we write down a model

$$Z_t = \alpha + \beta Z_{Mt} + \varepsilon_t$$

where $\mathbb{E}\varepsilon_t = 0, \mathrm{Cov}(Z_{Mt}, \varepsilon_t) = 0$, then the CAPM implies that $\alpha = 0$. To test CAPM is then to test the null hypothesis that

$$H_0 : \alpha = 0$$

against the alternative

$$H_1 : \alpha \neq 0.$$

We need to first estimate α and β. We take expectation on both sides of our econometric model and obtain

$$\mathbb{E}Z_t = \alpha + \beta \mathbb{E}Z_{Mt}.$$

Then we have

$$Z_t - \mathbb{E}Z_t = \beta(Z_{Mt} - \mathbb{E}Z_{Mt}) + \varepsilon_t.$$

Then we can estimate β by OLS, where the expectations can be replaced by their sample analogue:

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} Z_t,$$

$$\hat{\mu}_M = \frac{1}{T}\sum_{t=1}^{T} Z_{Mt},$$

$$\hat{\beta} = \frac{\sum_{t=1}^{T}(Z_t - \hat{\mu})(Z_{Mt} - \hat{\mu}_M)}{\sum_{t=1}^{T}(Z_{Mt} - \hat{\mu}_M)^2}.$$

Now we may estimate α as

$$\hat{\alpha} = \hat{\mu} - \hat{\beta}\hat{\mu}_M.$$

Under some regularity conditions, we have that under the null,

$$\sqrt{T}\hat{\alpha} \to_d \mathbb{N}\left(0, \left[1 + \frac{\mu_M^2}{\sigma_M^2}\right]\Sigma\right)$$

where $\mu_M = \mathbb{E}Z_{Mt}$, and $\Sigma = \mathbb{E}\varepsilon_t\varepsilon_t'$.

The asymptotic distribution contains unknown parameters, but we may estimate them by

$$\hat{\mu}_M = \frac{1}{T}\sum_{t=1}^{T} Z_{Mt},$$

$$\hat{\sigma}_M^2 = \frac{1}{T}\sum_{t=1}^{T}(Z_{Mt} - \hat{\mu}_M)^2,$$

and

$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T}(Z_t - \hat{\alpha} - \hat{\beta}Z_{Mt})(Z_t - \hat{\alpha} - \hat{\beta}Z_{Mt})'.$$

We use the Wald test statistic to test the null. The Wald test statistic is given by

$$W = \hat{\alpha}'(\text{Avar}(\hat{\alpha}))^{-1}\hat{\alpha} = T\left[1 + \frac{\mu_M^2}{\sigma_M^2}\right]^{-1}\hat{\alpha}'\Sigma^{-1}\hat{\alpha}.$$

Under the null, $W \sim \chi_N^2$.