On the Error Correction Model for Functional Time Series with Unit Roots

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Abstract

This paper studies the error correction model for functional time series with unit roots, which generalizes the vector error correction model for finite dimensional time series with unit roots. We unravel two important facts on the functional error correction model. First, any functional time series generated by an error correction model with a compact error correction operator has infinite dimensional unit roots. Second, the Granger's representation theorem continues to hold for the functional time series with unit roots in a form essentially identical to that for the finite dimensional error correction model.

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1 Introduction

The error correction model (ECM) has been widely used to analyze economic time series with common stochastic trends since the publication of the seminal papers by Granger (1981) and Engle and Granger (1987). In their analysis of a system of multiple economic time series, individual economic time series with stochastic trends are characterized as unit root processes, and the presence of common stochastic trends is interpreted as a consequence of cointegration representing longrun economic equilibria and reducing the number of unit roots in the system. In his study on the statistical inference of VARs with unit roots and cointegration, Johansen (1991) more formally derives the so-called Granger's representation theorem, which relates the ECM for time series with unit roots to its infinite order moving average representation in difference.

In this paper, we study the ECM of functional time series with unit roots, which will be referred to as the functional ECM. It was also studied recently by Beare and Seo (2015), which takes a quite different approach from ours as will be explained below. A functional time series is a time series of random elements taking values in an infinite dimensional Hilbert space of functions. See the monograph by Bosq (2000) for more detailed discussions on functional time series, and Kargin and Onatski (2008), Park and Qian (2012) and Hu et al. (2016) for their applications in forecasting and econometrics. Recently, Chang et al. (2016b) and Chang et al. (2016a) show that it is quite common to observe the presence of unit roots in functional time series data. In fact, they find some strong evidence of unit roots in many different functional time series data. Therefore, a question naturally arises whether the ECM is meaningfully defined and the Granger's representation theorem holds for the functional time series with unit roots.

In the functional ECM, the error correction term is specified by an operator, which we call the error correction operator. In this paper, we show that all compact error correction operators are necessarily of finite-rank. This implies that the unit root dimension of any functional time series generated by the ECM with a compact operator is infinite dimensional. No functional time series with finite dimensional unit roots may be generated through an ECM with a compact operator, or equivalently, no functional time series driven by a finite number of unit root processes allows for an ECM with a compact operator. A linear operator in a Hilbert space is compact if and only if it is the limit of a sequence of finite-rank operators, and therefore, we may approximate a linear operator in a Hilbert space with a finite dimensional matrix in any meaningful sense if and only if it is compact.

The functional ECM with a finite dimensional error correction operator and infinite dimensional unit roots is well defined and can be analyzed similarly as the ECM for finite dimensional time series. In particular, we demonstrate that the Granger's representation is possible for such a functional ECM and can be obtained analogously as in the finite dimensional case.

Our model and approach in the paper are different from those of Beare and Seo (2015) in two important aspects. While we mainly consider the functional ECM with a compact error correction operator, they derive their main theorem assuming that the error correction operator in their model has infinite dimensional range and therefore non-compact. Moreover, our assumptions and conclusions are contrastingly different from theirs. We impose an appropriate set of assumptions on the functional time series generated by a functional ECM to derive its infinite order moving average representation in difference, following the conventional approach to establish the Granger's representation theorem as in Johansen (1991, 1995) and Hansen (2005). On the other hand, Beare and Seo (2015) assume that the functional time series generated by a functional ECM has an infinite order moving average representation in difference and, under this assumption, they derive the restrictions on its longrun impact operator implied by the error correction operator.

2 Main Results

We denote by (f_t) a functional time series, where f_t for each $t = 1, 2, \ldots$ is a random element taking values in a separable Hilbert space H, and let (f_t) be generated as

$$\Delta f_t = \Lambda f_{t-1} + \sum_{k=1}^{p} \Gamma_k \Delta f_{t-k} + \varepsilon_t \tag{1}$$

where Δ is the usual difference operator, (ε_t) is a functional white noise, and Λ and Γ_k's are linear operators on H. Clearly, our model in (1) can be rewritten as

$$A(L) f_t = \varepsilon_t,$$

where L is the lag operator and

$$A(z) = A(1) + (1 - z) A_\Delta(z)$$

with

$$A(1) = -\Lambda \quad \text{and} \quad A_\Delta(z) = I + \Lambda - \sum_{k=1}^{p} \Gamma_k z^k.$$

The model in (1) extends the ECM for finite dimensional time series. Therefore, it will be referred to as the functional ECM. The error correction term of the functional ECM in (1) is given by the error correction operator Λ.

We assume that

Assumption 1 Λ is compact.

Assumption 2 There exists $\epsilon > 0$ such that $A(z)$ is invertible for all $|z| \leq 1 + \epsilon$ and $z \neq 1$.

Assumption 1 is not stringent. It is well known that a linear operator on a Hilbert space is compact if and only if it is given as the limit of a sequence of finite-rank operators. Therefore, we may approximate a linear operator on a Hilbert space with a finite dimensional matrix in any meaningful sense if and only if it is compact. Assumption 2 is necessary to ensure that the nonstationarity of (f_t) can be removed by differencing. It is also made in Johansen (1991, 1995) for his study on the finite dimensional ECM.

The following lemma gives a *necessary* condition for a functional time series (f_t) generated by the ECM in (1) to be integrated of order 1.

Lemma 1 Let Assumptions 1 and 2 hold. If (f_t) is I(1), then Λ has finite rank.

Proof of Lemma 1 To prove Lemma 1, note that, for $|z| \leq 1 + \epsilon$ and $z \neq 1$, we have

$$A^{-1}(z) = \frac{1}{z}\left[\frac{1-z}{z}\left(I - \sum_{k=1}^{p}\Gamma_k z^k\right) - \Lambda\right]^{-1}.$$

For (f_t) to be I(1), $A^{-1}(z)$ must have a simple pole at 1, which holds if and only if the generalized resolvent

$$\left[\lambda\left(I - \sum_{k=1}^{p}\Gamma_k\right) - \Lambda\right]^{-1}$$

has a simple pole at 0. It follows from Theorem 4.2 and the remarks on page 164 of Bart and Lay (1974) that this happens only if we can decompose H as

$$H = \mathcal{R}(\Lambda) \oplus \left(I - \sum_{k=1}^{p}\Gamma_k\right)\mathcal{N}(\Lambda)$$

where $\mathcal{R}(\Lambda)$ is closed. By Theorem 4.18 of Rudin (1991), as a compact operator, Λ must be of finite rank . □

Lemma 1 implies that if the error correction operator is compact and has infinite dimensional range space, then (f_t) can not be integrated of order one. We therefore assume that Λ has finite rank and write

$$\Lambda = \sum_{i=1}^{m} \lambda_i (u_i \otimes v_i) \tag{2}$$

with $\lambda_i \neq 0$ for $i = 1, \ldots, m$, where (u_i) and (v_i) are orthonormal bases of H and $\otimes$ is the tensor product on H. Subsequently, we let

$$U_m = \bigvee_{i=1}^{m} u_i \quad \text{and} \quad U_m^{\perp} = \bigvee_{i=m+1}^{\infty} u_i,$$

and similarly,

$$V_m = \bigvee_{i=1}^{m} v_i \quad \text{and} \quad V_m^{\perp} = \bigvee_{i=m+1}^{\infty} v_i,$$

where $\bigvee$ denotes (the closure of) the linear span of a set of functions in H. Furthermore, we define

$$P_m = \sum_{i=1}^{m} (u_i \otimes u_i) \quad \text{and} \quad P_m^{\perp} = \sum_{i=m+1}^{\infty} (u_i \otimes u_i),$$

and

$$Q_m = \sum_{i=1}^{m} (v_i \otimes v_i) \quad \text{and} \quad Q_m^{\perp} = \sum_{i=m+1}^{\infty} (v_i \otimes v_i),$$

so that P_m and $P_m^{\perp}$ are projections on U_m and $U_m^{\perp}$, and Q_m and $Q_m^{\perp}$ are projections on V_m and $V_m^{\perp}$, respectively. Finally, we denote by $\langle \cdot, \cdot \rangle$ the inner product defined on H.

To derive the Granger's representation theorem, we need to introduce an additional assumption.

Assumption 3 The linear operator $\Phi = P_m^{\perp} A_{\Delta}(1) Q_m^{\perp}$ is invertible as a map from $V_m^{\perp}$ to $U_m^{\perp}$.

Assumption 3 reduces to the condition in the Granger's representation theorem of Johansen (1991, 1995) where H is finite dimensional. It turns out that Assumption 3, together with that Λ has finite rank, is sufficient for the functional time series (f_t) to be I(1). If we write

$$\Phi = \sum_{i=m+1}^{\infty} \sum_{j=m+1}^{\infty} a_{ij} (u_i \otimes v_j),$$

the invertibility of Φ in Assumption 3 implies the existence of the linear operator

$$\Psi = \sum_{i=m+1}^{\infty} \sum_{j=m+1}^{\infty} b_{ij}(v_i \otimes u_j) \tag{3}$$

such that

$$\Phi\Psi = 1 \quad \text{and} \quad \Psi\Phi = 1$$

respectively on $U_m^{\perp}$ and $V_m^{\perp}$, from which it follows, in particular, that

$$U_m \cap V_m^{\perp} = \{0\} \quad \text{and} \quad U_m^{\perp} \cap V_m = \{0\}.$$

Under Assumption 3, V_m becomes the cointegrating space, as will be shown subsequently. Therefore, for any $v \in V_m$, $\big(\langle v, f_t \rangle\big)$ is a stationary time series. However, if $v \in U_m^{\perp}$, then $\langle v, \Delta f_t \rangle = \langle v, \varepsilon_t \rangle$ and $(\langle v, f_t \rangle)$ becomes a unit root process. This is the reason why $U_m^{\perp} \cap V_m = \{0\}$. We may similarly explain why we should have $U_m \cap V_m^{\perp} = \{0\}$.

Theorem 2 Let Assumptions 1, 2 and 3 hold. Then we may write

$$\Delta f_t = \Psi \varepsilon_t + \Delta g_t,$$

where

$$g_t = \sum_{k=0}^{\infty} \Pi_k \varepsilon_{t-k}$$

and Π_k's are absolutely summable in the operator norm.

Proof of Theorem 2 Define

$$\begin{aligned} C(z) &= A(z)\left(Q_m + \frac{1}{1-z} Q_m^{\perp}\right) \\ &= \Big(-\Lambda + (1-z)A_{\Delta}(z)\Big)\left(Q_m + \frac{1}{1-z} Q_m^{\perp}\right) \\ &= \Big(-\Lambda + (1-z)P_m A_{\Delta}(z) Q_m\Big) + (1-z)P_m^{\perp} A_{\Delta}(z) Q_m + P_m A_{\Delta}(z) Q_m^{\perp} + P_m^{\perp} A_{\Delta}(z) Q_m^{\perp}. \end{aligned}$$

It follows that

$$C(1) = -\Lambda + P_m A_{\Delta}(1) Q_m^{\perp} + P_m^{\perp} A_{\Delta}(1) Q_m^{\perp},$$

which is invertible as a mapping from $V_m^\perp$ to $U_m^\perp$ under Assumption 3, and we have

$$C^{-1}(1) = -\left(\sum_{i=1}^{m} \frac{1}{\lambda_i}(v_i \otimes u_i)\right)\left(1 - A_\Delta(1)\Psi\right) + \Psi$$

upon noticing that $\Phi = P_m^\perp A_\Delta(1) Q_m^\perp$. In particular, under both Assumptions 1 and 2, $C^{-1}(z)$ is analytic for $|z| < 1$.

However, for $|z| \leq 1$ and $z \neq 1$, it follows that

$$A^{-1}(z) = \left(Q_m + \frac{1}{1-z} Q_m^\perp\right) C^{-1}(z).$$

Therefore, if we let

$$B(z) = (1-z)A^{-1}(z) = \left((1-z)Q_m + Q_m^\perp\right) C^{-1}(z),$$

then $B(z)$ is analytic for $|z| < 1$ under Assumptions 2 and 3, and we may easily deduce that

$$B(1) = \Psi,$$

and that

$$B(z) - B(1) = (1-z)\left(Q_m C^{-1}(z) + Q_m^\perp C_\Delta^{-1}(z)\right),$$

where

$$C_\Delta^{-1} = \frac{C^{-1}(z) - C^{-1}(1)}{1-z},$$

which is analytic for $|z| < 1$. The stated result now follows readily from

$$\Delta f_t = B(L)\varepsilon_t = \left[B(1) + (1-L)\left(Q_m C^{-1}(L) + Q_m^\perp C_\Delta^{-1}(L)\right)\right]\varepsilon_t,$$

and the proof is complete. □

Theorem 2 is completely analogous to the Granger's representation theorem in Johansen (1991, 1995). In fact, it is straightforward to show that Theorem 2 reduces to Theorem 4.1 in Johansen (1991) if H is finite dimensional. It follows, in particular, from Theorem 2, together with the definitions of Λ and Ψ respectively in (2) and (3), that

$$\Lambda\Psi = \Psi\Lambda = 0,$$

which is the essence of the Granger's representation theorem. If we set $f_0 = g_0$, then

$$f_t = \Psi \sum_{k=1}^{t} \varepsilon_k + g_t$$

for all $t = 1, 2, \ldots$. For all $v \in V_m$, we have

$$\langle v, f_t \rangle = \langle v, g_t \rangle$$

and therefore $\big(\langle v, f_t \rangle\big)$ is stationary. On the other hand, for all $v \in V_m^\perp$, we have

$$\langle v, f_t \rangle = \left\langle v, \Psi \sum_{k=1}^{t} \varepsilon_k \right\rangle + \langle v, g_t \rangle = \left\langle \Psi^* v, \sum_{k=1}^{t} \varepsilon_k \right\rangle + \langle v, g_t \rangle,$$

where Ψ^* is the adjoint operator of Ψ and we have $\Psi^* v \neq 0$ for any $v \in V_m^\perp$ under Assumption 3, which implies that $\big(\langle v, f_t \rangle\big)$ is a unit root process. Therefore, V_m is the cointegrating subspace of H. In particular, we have a finite dimensional cointegrating space and infinite dimensional unit roots, if (f_t) is defined in an infinite dimensional Hilbert space H.

3 Concluding Remark

We show that the functional ECM with a compact error correction operator necessarily has a finite-rank error correction term and infinite dimensional unit roots. Moreover, we establish the Granger's representation theorem for the functional time series with unit roots analogously as for the finite dimensional ECM. Our assumptions are minimal, and reduce to the standard conditions imposed for the finite dimensional ECM where the underlying functional time series become degenerate and finite dimensional.

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